Densità di Weibull (α, λ)

$$f(x) = \begin{cases} \lambda \alpha x^{\alpha-1} e^{-\lambda x^{\alpha}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} \lambda \underline{\alpha x^{\alpha-1}} e^{-\lambda x^{\alpha}} \underline{dx}$$

$$= \int_0^{+\infty} \lambda e^{-\lambda y}\, dy = 1$$

$x^{\alpha} = y$

$\alpha x^{\alpha-1} dx = dy$

Per $\alpha = 1$ si ritrova l'esponenziale di par. λ

$X \sim f$ X = durata di uno strumento

$$t \mapsto P(X > t+s \mid X > t) = P(X > s)$$

Se $X \sim \mathcal{E}(\lambda)$ la funzione

$$t \longmapsto P(X > t+s \mid X > t)$$

non dipende da t (costante in t

Se c'è usura, devo richiedere

che $t \longmapsto P(X > t+s \mid X > t)$

sia decrescente

$$P(X > t+s \mid X > t) =$$

$$= \frac{P(X > t+s, X > t)}{P(X > t)} = \frac{P(X > t+s)}{P(X > t)} =$$

$$= \frac{\int_{t+s}^{\infty} \lambda \alpha x^{\alpha-1} e^{-\lambda x^\alpha} dx}{\int_{t}^{+\infty} \lambda \alpha x^{\alpha-1} e^{-\lambda x^\alpha} dx} =$$

$x^\alpha = y$

$$= \frac{\int_{(t+s)^\alpha}^{+\infty} \lambda e^{-\lambda y} dy}{\int_{t^\alpha}^{+\infty} \lambda e^{-\lambda y} dy} = \frac{-e^{-\lambda y}\Big|_{(t+s)^\alpha}^{+\infty}}{-e^{-\lambda y}\Big|_{t^\alpha}^{+\infty}} =$$

$$= \frac{e^{-\lambda(t+s)^\alpha}}{e^{-\lambda t^\alpha}} = e^{-\lambda(t+s)^\alpha + \lambda t^\alpha}$$

$$t \longmapsto e^{-\lambda \left(\underbrace{(t+s)^\alpha - t^\alpha}\right)} \quad \text{è decresc}$$

$$t \longmapsto (t+s)^\alpha - t^\alpha = g(t) \text{ è crescente}$$

$$g'(t) = \alpha (t+s)^{\alpha-1} - \alpha t^{\alpha-1} \geq 0$$

$$(t+s)^{\alpha-1} \geq t^{\alpha-1} \implies \alpha - 1 \geq 0$$

$$(\alpha-1) \log(t+s) \geq (\alpha-1) \log t$$

$$\alpha \geq 1$$

$$\alpha < 1$$

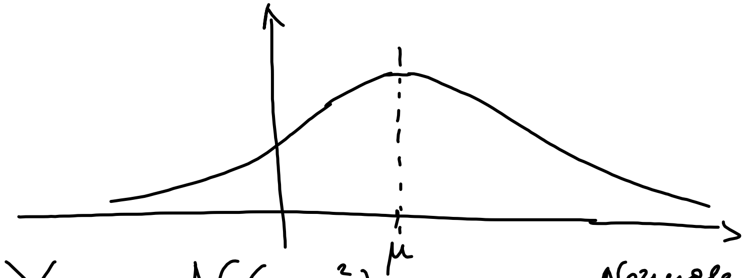
Densità (o leggi) gaussiane
o Normali

Def. Si dice che X ha legge gaussiana di parametri μ e σ^2 ($\mu \in \mathbb{R}$, $\sigma^2 > 0$) se ha la seguente densità

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\boxed{\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \sqrt{2\pi\sigma}}$$



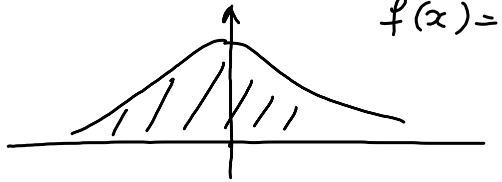
$X \sim N(\mu, \sigma^2)$

Normale standard

$\mu = 0 \;,\; \sigma^2 = 1$ $\qquad N(0,1)$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \qquad x \in \mathbb{R}}$$

$f(x) = f(-x)$



Si dice che X è una v. a. simmetrica se X e $-X$ hanno la stessa legge.

Se X ha densità f pari ($f(x) = f(-x)$), allora X è simmetrica.

Funzione di ripart. di $X = \int_{-\infty}^{t} f(x)\,dx$

Funzione di ripart. di $-X$

$$P(-X \le t) = P(X \ge -t) =$$

$$= \int_{-t}^{+\infty} f(x)\,dx \overset{f \text{ è pari}}{=} \int_{-t}^{+\infty} f(-x)\,dx$$

$$-x = y$$
$$dx = -dy$$

$$= -\int_{t}^{-\infty} f(y)\,dy = \int_{-\infty}^{t} f(y)\,dy$$

$$= P(X \le t)$$

$$\cancel{X = -X} \quad \text{no!}$$

<u>Proposizione.</u> Sia $X \sim N(m, s^2)$

Allora la v. a. $Y = \underline{aX+b} \sim N(am+b, a^2s^2)$ $a \neq 0$

Dim.

$$P(Y \leq t) = P(aX + b \leq t) = \quad a>0$$

$$= P(aX \leq t-b) =$$

$$= P\left(X \leq \frac{t-b}{a}\right) =$$

$$= \int_{-\infty}^{\frac{t-b}{a}} f(x)\,dx = \int_{-\infty}^{\frac{t-b}{a}} \frac{1}{\sqrt{2\pi}\,s} e^{-\frac{(x-m)^2}{2s^2}}\,dx$$

$$P(Y \leq t) = \int_{-\infty}^{\frac{t-b}{a}} \frac{1}{\sqrt{2\pi}\,s} e^{-\frac{(x-m)^2}{2s^2}}\,dx$$

$$-\frac{\left(\frac{t-b}{a} - m\right)^2}{2s^2}$$

$$\frac{d}{dt} P(Y \leq t) = \frac{1}{\sqrt{2\pi}\,s} e^{-\frac{\left(\frac{t-b-am}{a}\right)^2}{2s^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi}\,(as)} e^{-\frac{(t-\overbrace{[b+am]}^{\mu=})^2}{2\underbrace{a^2s^2}_{\sigma^2}}}$$

$$= \frac{1}{\sqrt{2\pi}\,\underbrace{(as)}_{\sigma}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \sim N(am+b, a^2s^2)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e$$

$$X \sim N(m, s^2) \Rightarrow$$

$$Y = aX$$

$$P(Y \leq t) = P(aX \leq t-b) \qquad a<0$$

$$= P\left(X \geq \frac{t-b}{a}\right) =$$

$$= \int_{\frac{t-b}{a}}^{+\infty} \frac{1}{\sqrt{2\pi}\,s} e^{-\frac{(x-m)^2}{2s^2}} dx$$

$$= -\int_{+\infty}^{\frac{t-b}{a}} \cdots\cdots \, dx$$

$$\frac{d}{dt} P(Y \leq t) = \underbrace{-\frac{1}{\sqrt{2\pi}(as)} e^{-\frac{(t-[am+b])^2}{2a^2s^2}}}$$

$$= \frac{1}{\sqrt{2\pi}\,|as|} e^{-\frac{(t-[am+b])^2}{2(as)^2}}$$

$$\sim N(am+b, (as)^2)$$

Corollario (i) Se $X \sim N(0,1)$

Allora $Y = \underline{\sigma X + \mu} \sim N(\mu, \sigma^2)$

(ii) Se $X \sim \underline{N(\mu, \sigma^2)}$, allora

$$Y = \frac{X-\mu}{\sigma} \sim N(0,1)$$

Dim.

(i) $$Y \sim N(\sigma \cdot \overset{m}{0} + \mu, 1 \cdot \sigma^{\overset{s^2}{2}}) =$$

$$= N(\mu, \sigma^2)$$

(ii) $$Y = \frac{X-\mu}{\sigma} = \underbrace{\frac{1}{\sigma}}_{a} X + \underbrace{\left(\frac{-\mu}{\sigma}\right)}_{b}$$

$$Y \sim N(\cancel{a}\underbrace{m}_{=\mu} + b, a^2 \underbrace{s^2}_{=\sigma^2}) =$$

$$= N\Big(\underbrace{\frac{1}{\sigma} \cdot \mu + \left(\frac{-\mu}{\sigma}\right)}, \underbrace{\left(\frac{1}{\sigma}\right)^2} \sigma^2\Big) = N(0,1)$$

Sia $Y \sim N(0,1)$.

$\underline{P(Y \le t)}$

Sia $X \sim N(\mu, \sigma^2)$

$$X \sim N(\mu, \sigma^2) \qquad Y = \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$\sigma > 0$$

$$P(\underline{X \le t}) =$$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{t-\mu}{\sigma}\right) = P\left(Y \le \frac{t-\mu}{\sigma}\right)$$

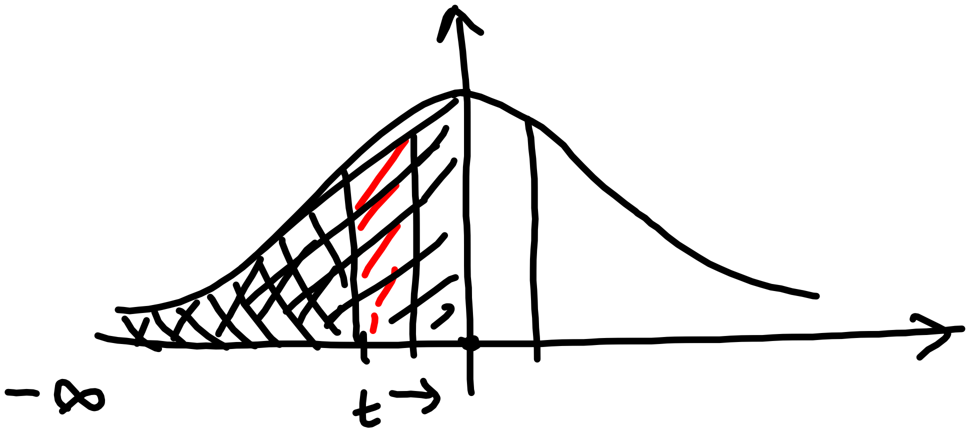
$$=$$

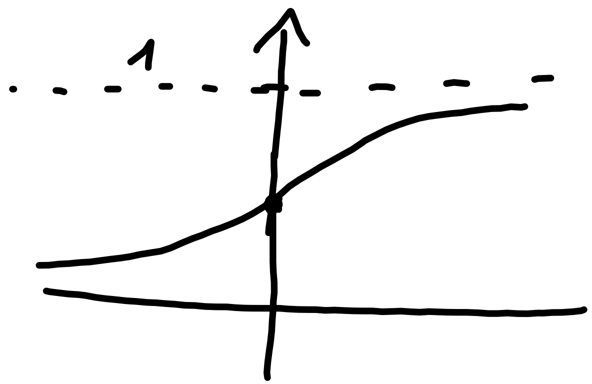
Sia $Y \sim N(0,1)$

Si pone

$$\phi(t) = P(Y \le t) = \boxed{\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}$$

$$\Phi(t) = \int_{-\infty}^{t} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}\, dx \qquad \int e^{-x^2}\, dx$$

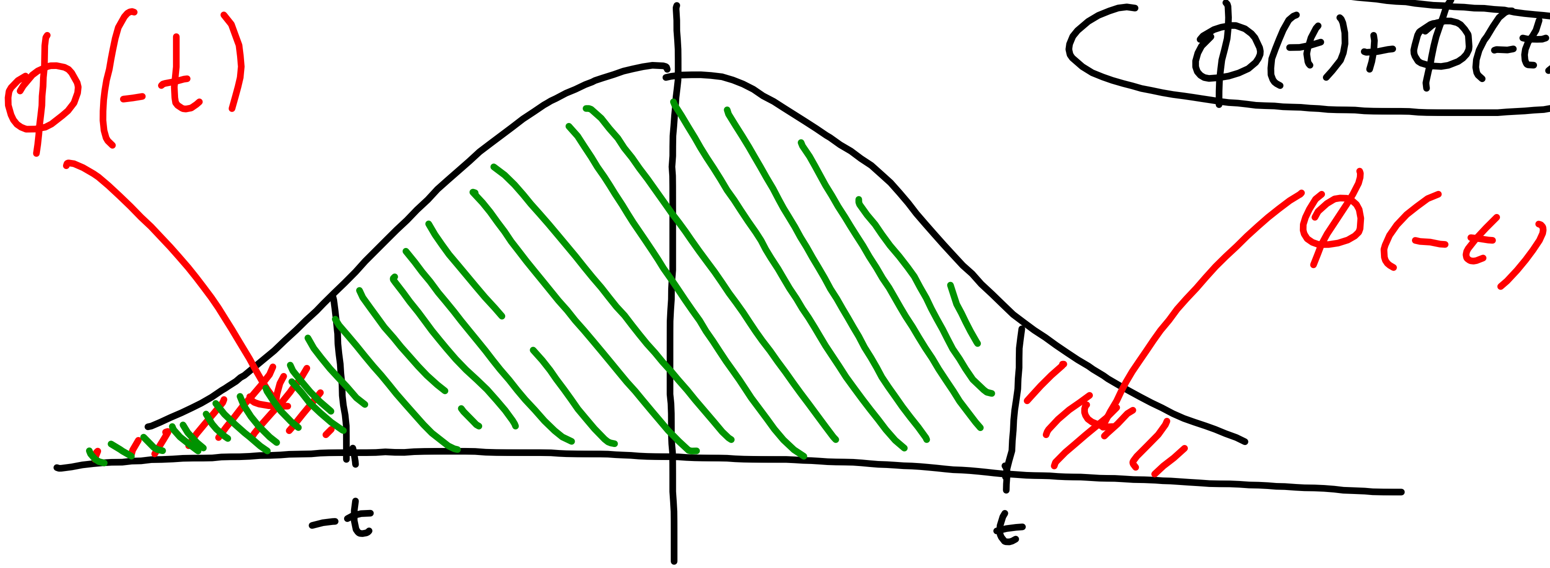


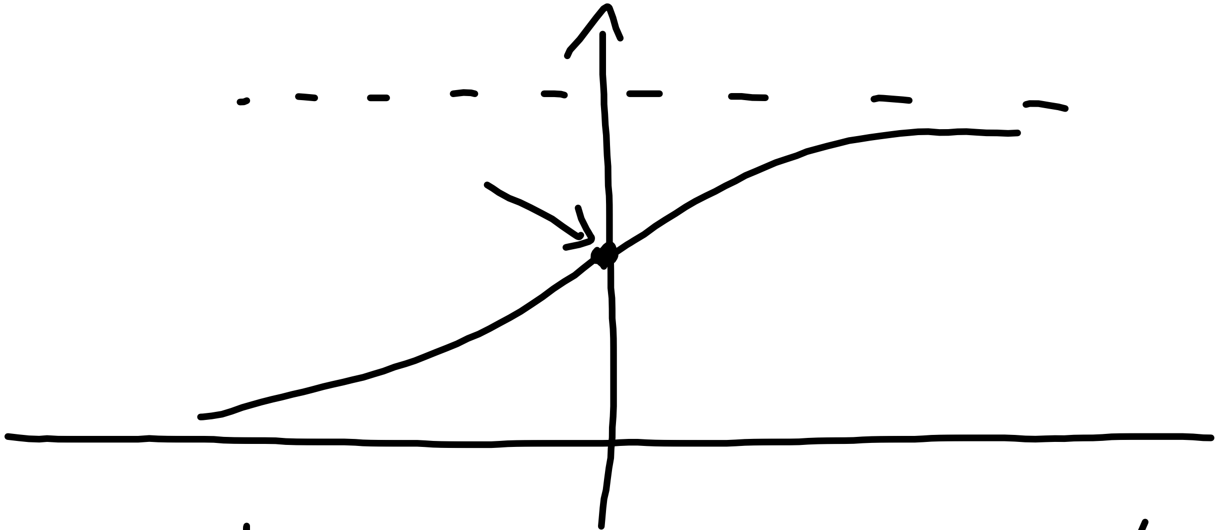


$$\Phi(0) = \frac{1}{2}$$

$$\Phi(t) + \Phi(-t) = 1$$

$\Phi(t)+\Phi(-t)=1$





$\Phi(t)+\Phi(-t)=1$ $\quad$ $\Phi(0)+\Phi(0)=1$

Sia $X \sim N(\mu, \sigma^2)$. Allora

$$E[X] = \mu$$

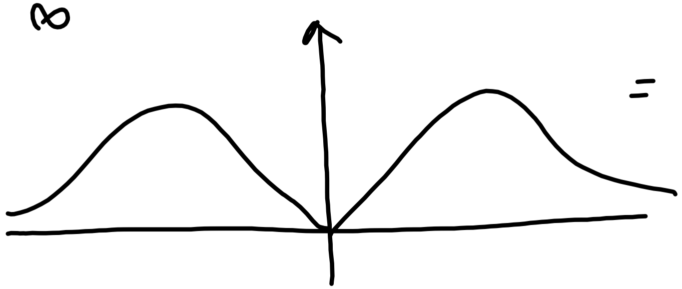
$$\mathrm{Var}\, X = \sigma^2$$

Caso della $N(0,1)$

Sia $X \sim N(0,1)$

$$E[|X|] = \int_{-\infty}^{+\infty} |x| f(x)\, dx$$

$$\int_{-\infty}^{+\infty} |x|\, f(x)\, dx = \int_{-\infty}^{+\infty} |x| \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$



$$= 2\int_{0}^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} x\, e^{-x^2/2}\, dx = \frac{2}{\sqrt{2\pi}} \left[-e^{-\frac{x^2}{2}} \right]_{0}^{+\infty}$$

$$= \frac{2}{\sqrt{2\pi}}\, 1 = \frac{2}{\sqrt{2\pi}}$$

$$\int x^m e^{-\frac{x^2}{2}}\, dx \qquad m=0 \quad \int e^{-x^2/2}\, dx$$

$m=1$

se m dispari si

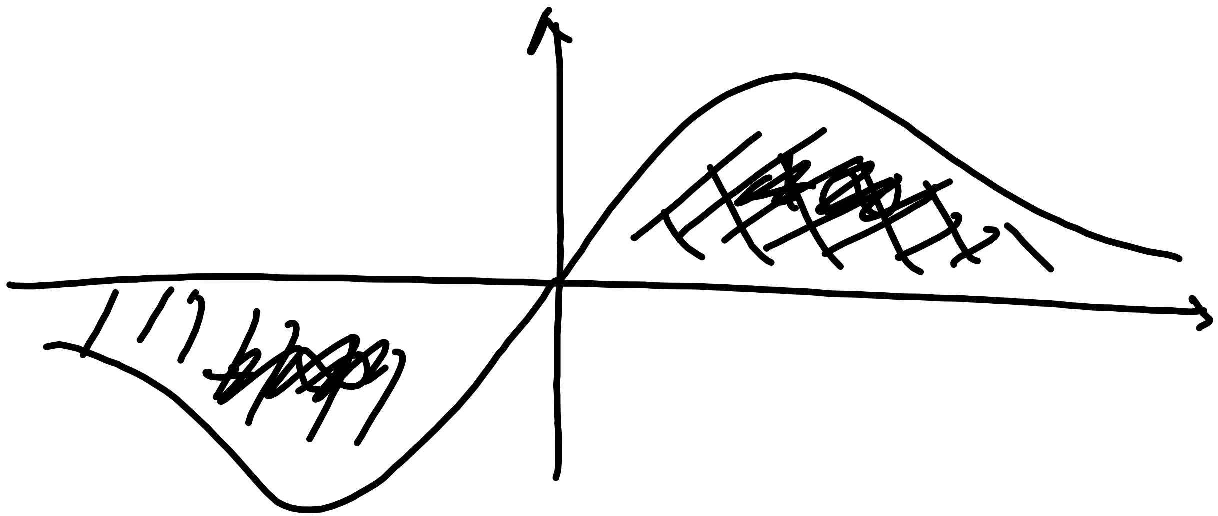
sa scrivere la primitiva

no se m è pari

$$E[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx =$$

$$f(x) = -f(-x)$$

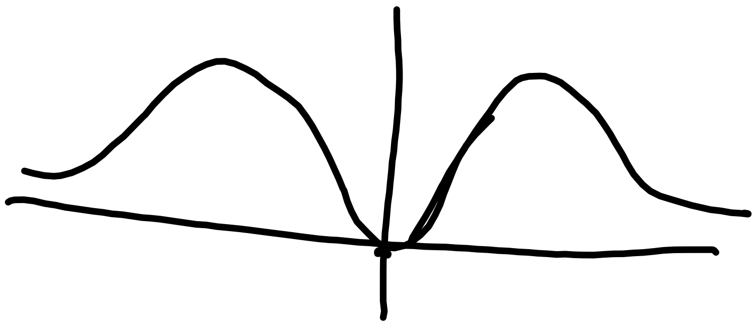
$$= \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 0$$



$$\mathrm{Var}\,X = \underbrace{E[X^2]}_{=1} - \underbrace{E^2[X]}_{=0} = 1$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} \underbrace{x^2}_{\downarrow} \underbrace{e^{-x^2/2}}_{\downarrow 0}\,dx$$



Se $X \sim N(0,1)$, allora

$E[X]=0$, $\mathrm{Var}\,X = 1$

Se $X \sim N(\mu, \sigma^2)$, allora

$E[X]=\mu$, $\mathrm{Var}\,X = \sigma^2$

$$Y = \frac{X-\mu}{\sigma} \sim \underline{\underline{N(0,1)}} \Rightarrow X = \sigma Y + \mu$$

$$E[X] = E[\sigma Y + \mu] = \sigma \underbrace{E[Y]}_{=0} + \mu = \mu$$

$$\mathrm{Var}\,X = \mathrm{Var}(\sigma Y + \mu) =$$

$$= \mathrm{Var}(\sigma Y) = \sigma^2 \underbrace{\mathrm{Var}\,Y}_{=1} = \sigma^2$$

Teorema. Siano $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ $\Big\}$ indipend.

Allora la v.a.

$$Z = X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$E[Z] = E[X+Y] = E[X] + E[Y] = \mu_1 + \mu_2$$

$$\mathrm{Var}\, Z = \mathrm{Var}(X+Y) = \mathrm{Var}\, X + \mathrm{Var}\, Y = \sigma_1^2 + \sigma_2^2$$

$$X \sim N(m, s^2) \Rightarrow Y = aX + b \sim N(am+b, a^2 s^2)$$

$$E[Y] = E[aX+b] = aE[X] + b = am + b$$

$$\mathrm{Var}\, Y = \mathrm{Var}(aX+b) = a^2\, \mathrm{Var}\, X = a^2 s^2$$

$X \sim N(0,1)$

$2X \sim N(0,4)$ non sono indip.!

$aX+b$

$X+2X \sim N(0,5)$ no!

$= 3X \sim N(0,9)$

Siano $X_1, \dots, X_n$ n v.a. indipend. (esercizio)

tutte con legge $N(\mu, \sigma^2)$.

Calc. la legge di

$$\frac{X_1+\dots+X_n}{n} = \overline{X} = \text{media campiona}$$

$$Y = X_1+\dots+X_n \sim N(n\mu, n\sigma^2)$$

$$\overline{X} = \frac{X_1+\dots+X_n}{n} = \frac{1}{n}(Y) \sim N\left(\frac{n\mu}{n}, \frac{n\sigma^2}{n^2}\right) = N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$X \sim N(\mu, \sigma^2) \to \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$\frac{X - E[X]}{\sqrt{\operatorname{Var} X}} = Y$$

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} = \sqrt{n}\,\underbrace{\frac{\overline{X}-\mu}{\sigma}} \sim N(0,1)$$

$$= \frac{\frac{X_1+\dots+X_n}{n} - \mu}{\sigma}\sqrt{n} =$$

$$= \frac{\frac{X_1+\dots+X_n - n\mu}{n}\cdot\sqrt{n}}{\sigma} =$$

$$= \frac{X_1+\dots+X_n - n\mu}{\sigma\sqrt{n}} = \sqrt{n}\,\frac{\overline{X}-\mu}{\sigma}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^2 \, e^{-\frac{x^2}{2}} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underline{\underline{x}} \; \underbrace{\left(x e^{-x^2/2}\right)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\underline{-e^{-\frac{x^2}{2}}} \cdot \underline{\underline{x}} \,\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -e^{-\frac{x^2}{2}} \, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(0 + \int_{-\infty}^{+\infty} e^{-x^2/2} \, dx \right)$$

$$= \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} \, dx = 1$$